# SAMPLE EXAMINATION III

## Answers to Multiple-Choice Questions

1. The area of the region is represented by $\int_1^3 (3x^2 + 2x)\, dx$.

   $\int_1^3 (3x^2 + 2x)\, dx = x^3 + x^2\Big|_1^3 = (27 + 9) - (1 + 1) = 36 - 2 = 34$

   The correct choice is (B).

2. Rewrite the function without the absolute value signs:

   $$f(x) = \begin{cases} -x, & x < 0 \\ 0, & x = 0 \\ x, & x > 0 \end{cases}$$

   Then
   $$\begin{aligned} \int_{-4}^{2} f(x)\, dx &= \int_{-4}^{0} (-x)\, dx + \int_{0}^{2} x\, dx \\ &= -\frac{x^2}{2}\Big|_{-4}^{0} + \frac{x^2}{2}\Big|_{0}^{2} \\ &= (0 - (-8)) + (2 - 0) \\ &= 10 \end{aligned}$$

   The correct choice is (D).

3. The units of a definite integral or a Riemann sum are the units of the dependent variable, $C(s)$, multiplied by the units of the independent variable, $ds$, in this case

$(gallons/mile) \times (miles/hour) = gallons/hour$

The correct choice is (B).

4. Given: $\frac{dr}{dt} = 2$

Find: $\frac{dV}{dt}$ when $r = 10$

Since the volume of a sphere is $V = \frac{4}{3}\pi r^3$, differentiate both sides of the equation with respect to $t$.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

or $\frac{dV}{dt} = 4\pi(10)^2 (2) = 800\pi$

The correct choice is (D).

5. Since the laser beam changes direction at $t = 2$, the total distance traveled by the laser beam is calculated using two integrals. For motion to the left: $\left|\int_1^2 (t^2 - 4)\, dt\right|$, and for motion to the right: $\int_2^3 (t^2 - 4)\, dt$.

$$\left|\int_1^2 (t^2 - 4)\, dt\right| = \left|\left(\frac{t^3}{3} - 4t\right)\Big|_1^2\right| = \left|\left(\frac{8}{3} - 8\right) - \left(\frac{1}{3} - 4\right)\right| = \frac{5}{3}$$

$$\int_2^3 (t^2 - 4)\, dt = \left(\frac{t^3}{3} - 4t\right)\Big|_2^3 = \left(\frac{27}{3} - 12\right) - \left(\frac{8}{3} - 8\right) = \frac{7}{3}$$

Therefore the total distance traveled by the laser beam is 4 feet.

The correct choice is (A).

6. By the Fundamental Theorem of Calculus: $\int_1^4 f'(x)\, dx = f(4) - f(1) = 2 - (-5) = 7$

The correct choice is (D).

7. The given limit is the derivative of $g(x)$ at $x = 3$. Since $g'(3)$ is negative, the function must be decreasing at $x = 3$.

The correct choice is (A).

8. Let $u = x^2 + 1$

$du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(x^2+1)\Big|_1^3 = \frac{1}{2}(\ln 10 - \ln 2)$$

Since $\ln 10 - \ln 2 = \ln\left(\frac{10}{2}\right) = \ln 5$, therefore, $\int_1^3 \frac{x}{x^2+1}dx = \frac{1}{2}\ln 5.$

The correct choice is (C).

9. Using the properties of logarithms and the Chain Rule:

$$\begin{aligned}\frac{d}{dx}\ln\left(\frac{1}{x^2-1}\right) &= \frac{d}{dx}(\ln 1 - \ln(x^2-1))\\ &= \frac{d}{dx}(-\ln(x^2-1))\\ &= -\frac{2x}{x^2-1} = \frac{2x}{1-x^2}\end{aligned}$$

The correct choice is (A).

10. Method 1: The average rate of change is given by $\frac{f(b)-f(a)}{b-a} = \frac{e^{(9)}-e^{(9)}}{3-(-3)} = 0$. The instantaneous rate of change is the derivative $f'(x) = 2xe^{(x^2)}$. Solving $2xe^{(x^2)} = 0$ gives only one value of $x = 0$.

Method 2: The instantaneous rate of change is given by the derivative of $f$: $f'(x) = 2xe^{(x^2)}$. Since the derivative is always increasing $\left(f''(x) = (4x^2+2)\,e^{(x^2)} > 0\right)$ it will equal every real number exactly once. Therefore whatever the average value is, it will equal it once.

The correct choice is (B).